# §14.5 The Chain Rule

∗The Chain Rule

$$\begin{cases} z(t) = f(x(t), y(t)) \Rightarrow \dfrac{dz}{dt} = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt} \\ u(t) = f(x(t), y(t), z(t)) \Rightarrow \dfrac{dz}{dt} = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt} + f_z \dfrac{dz}{dt} \end{cases}$$

**Example 1**：

$$z = f(x, y) = x^2 + xy + y^2, x = \cos t, y = \sin t.$$

**Solution**：

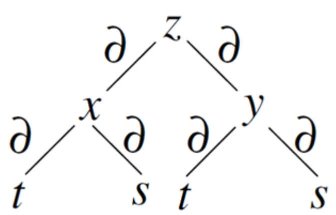
$$z = f(x, y) = x^2 + xy + y^2$$

$$z = \cos^2 t + \cos t \sin t + \sin^2 t$$

$$\begin{aligned} \frac{dz}{dt} &= 2\cos t(-\sin t) + (-\sin t)\sin t + \cos t \cos t + 2\sin t \cos t \\ &= (2\cos t + \sin t)(-\sin t) + (\cos t + 2\sin t)\cos t \\ &= (2x + y)\frac{dx}{dt} + (x + 2y)\frac{dy}{dt} = f_x\left(\frac{dx}{dt}\right) + f_y\left(\frac{dy}{dt}\right). \end{aligned}$$

∗Tree Diagram

$z = (t, s) = f(x(t, s), y(t, s))$




$$\Rightarrow \begin{cases} \dfrac{\partial z}{\partial t} = f_x \dfrac{\partial x}{\partial t} + f_y \dfrac{\partial y}{\partial t} \\ \dfrac{\partial z}{\partial s} = f_x \dfrac{\partial x}{\partial s} + f_y \dfrac{\partial y}{\partial s} \end{cases}$$

∗Implicit Differentiation

If $f(x, y, z) = 0$ and $z = g(x, y)$

$\Rightarrow f(x, y, g(x, y)) = 0$

$\Rightarrow f_x + f_y \dfrac{dy}{dx} + f_z \dfrac{\partial z}{\partial x} = 0$

Since $\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{\partial z}{\partial x} = -\dfrac{f_x}{f_z}$.

Similarly $\dfrac{\partial z}{\partial y} = -\dfrac{f_y}{f_z}$.

**Example 2**：

$u = x^4 y + y^2 z^2, x = rse^t, y = rs^2 e^{-t}, z = r^2 s \sin t.$

Find $\left.\dfrac{\partial u}{\partial s}\right|_{\substack{r=2\\ s=1\\ t=0}}$.

**Solution**：

$$\left.\frac{\partial u}{\partial s}\right|_{\substack{r=2\\ s=1\\ t=0}} = 4x^3 y\left(re^t\right) + \left(x^4 + 2yz^2\right)\left(2sre^{-t}\right) + 2y^2 z\left(r^2 \sin t\right)$$

$$= 128 + 64 = 192$$

**Example 3**：

$x^3 + y^3 + z^3 + 6xyz = 1$, find $\dfrac{\partial z}{\partial x}$.

**Solution**：

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0.$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-x^2 - 2yz}{z^2 + 2xy}.$$

**Example 4**：

Let $f$ has continuous $2^{\text{nd}}$ – order partial derivatives.

$z = f(x, y), \begin{cases} x = r^2 + s^2 \\ y = 2rs \end{cases}$, find $\dfrac{\partial^2 z}{\partial r^2}$.

**Solution**：

$$\frac{\partial z}{\partial r} = f_x 2r + f_y 2s$$

$$\frac{\partial^2 z}{\partial r^2} = \left(f_{xx} 2r + f_{xy} 2s\right)2r + 2f_x + \left(f_{yx} 2r + f_{yy} 2s\right)2s$$

$$= 4r^2 f_{xx} + 8rsf_{xy} + 4s^2 f_{yy} + 2f_x$$

$(f_{xy} = f_{yx})$

**Example 5**：

$PV = 8.31T$. Find $\frac{dP}{dt}$ when $T = 300\ K$ and is increasing at a rate of $0.1K/s$,

and $V = 100L$ and is increasing at a rate of $0.2L/s$.

**Solution**：

$$P=\frac{8.31T}{V}=\frac{8.31\times 300}{100}=24.93$$

$$\frac{dP}{dt}V+P\frac{dV}{dt}=8.31\frac{dT}{dt} \qquad (1)$$

$$\frac{dT}{dt}=0.1, \frac{dV}{dt}=0.2, T=300, V=100, P=24.93$$

代入(1)式求$\frac{dP}{dt}$.

**Example 6**：

$$z=f(x,y), \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

(i). Find $\frac{\partial^2 z}{\partial r^2}, \frac{\partial^2 z}{\partial \theta^2}$

(ii). Prove that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}+\frac{1}{r}\frac{\partial z}{\partial r}$.

**Solution**：

(i). $\frac{\partial z}{\partial r}=f_x\cos\theta+f_y\sin\theta$

$$\frac{\partial^2 z}{\partial r^2}=\left(f_{xx}\cos\theta+f_{xy}\sin\theta\right)\cos\theta+\left(f_{yx}\cos\theta+f_{yy}\sin\theta\right)\sin\theta$$
$$=f_{xx}\cos^2\theta+2f_{xy}\sin\theta\cos\theta+f_{yy}\sin^2\theta$$

$$\frac{\partial z}{\partial \theta}=f_x(-r\sin\theta)+f_y r\cos\theta$$

$$\frac{\partial^2 z}{\partial \theta^2}=\left[f_{xx}(-r\sin\theta)+f_{xy}r\cos\theta\right](-r\sin\theta)$$
$$+f_x(-r\cos\theta)+f_y(-r\sin\theta)+\left[f_{yx}(-r\sin\theta)+f_{yy}r\cos\theta\right]r\cos\theta.$$
$$=r^2\sin^2\theta f_{xx}-2r^2\cos\theta\sin\theta\, f_{xy}+r^2\cos^2\theta f_{yy}-r\cos\theta f_x-r\sin\theta f_y.$$

(ii). 可由(i)得.

**Example 7**：

If $g(s, t) = f(s^2 - t^2, t^2 - s^2)$ and $f$ is differentiable. Show that $g$ satisfies

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

**Proof**.

$$\frac{\partial g}{\partial s} = 2sf_x - 2sf_y$$

$$\frac{\partial g}{\partial t} = -2tf_x + 2tf_y$$

$$\Rightarrow t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$