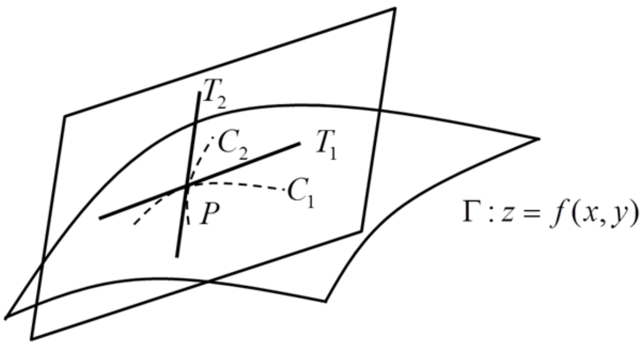
# §14.4 Tangent Planes and Linear Approximation

＊切線方程式




過在曲面$\Gamma$上的$P$點之切面

設此切面方程式為

$A(x-x_0)+B(y-y_0)+(z-z_0)=0$   (1)

Let $x = x_0$, then

$B(y-y_0)+(z-z_0)=0$ 為過 $P$ 點和曲線 $C_1 : z = f(x_0, y)$ 相切的直線方程式 $T_1$.

$\Rightarrow f_y(x_0, y_0) = T_1$的斜率$=-B$.

Similarly, $f_x(x_0, y_0) = T_2$的斜率$=-A$.

＊公式：

(i). 切面的法向量為

$$\langle A, B, 1\rangle = \langle -f_x(x_0, y_0), f_y(x_0, y_0), 1\rangle$$

(ii). 切面方程式為

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

**Example 1**：

$z = 2x^2 + y^2$. 找過 $P(1,1,3)$的切面方程式.

**Solution**：

$f_x(x, y) = 4x \Rightarrow f_x(1,1) = 4.$

$f_y(x, y) = 2y \Rightarrow f_y(1,1) = 2.$

Tangent plane: $4(x-1)+2(y-1) = z-3.$

＊Linear approximation of $z = f(x, y)$ at $(a, b)$.

(以切平面來逼近曲面)

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$

**Example 2**：

$$z=\sqrt{x^2+y^2} \text{ at } (3,4). \text{ Find } z(3.01,3.99).$$

**Solution**：

$$f_x(x,y)=\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2x) \Rightarrow f_x(3,4)=\frac{3}{5}$$

$$f_y(x,y)=\frac{1}{2}\left(x^2+y^2\right)^{-\frac{1}{2}}(2y) \Rightarrow f_y(3,4)=\frac{4}{5}$$

$$\Rightarrow z=5+\frac{3}{5}(0.01)+\frac{4}{5}(-0.01)=4.998\approx\sqrt{(3.01)^2+(3.99)^2}.$$

＊Differentiable of *f* at ( *a*, *b* )：

**Definition**：

If $\Delta z = f(a+\Delta x, b+\Delta y)-f(a,b)$

$$= f_x(a,b)\Delta x + f_y(a,b)\Delta y + \underbrace{\varepsilon_1\Delta x+\varepsilon_2\Delta y}_{誤差},.$$

then, $\varepsilon_1,\varepsilon_2 \to 0$ as $(\Delta x,\Delta y)\to(0,0)$ (切面≈曲面)

**Example 3**：

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq 0\\ 0 & ,(x,y)=0\end{cases}$$

**Solution**：

$f_x(0,0)=f_y(0,0)=0$

Tangent plane through $(0,0): z=0.$

But $f(x,y)=\frac{1}{2}$ whenever $x=y\neq 0.$

$\Rightarrow f$ is not differentiable at $(a,b).$

**Theorem**：

$f$ is differentiable at ( $a$, $b$ ) provided that

(i). $f_x$, $f_y$ exist near ( $a$, $b$ ).

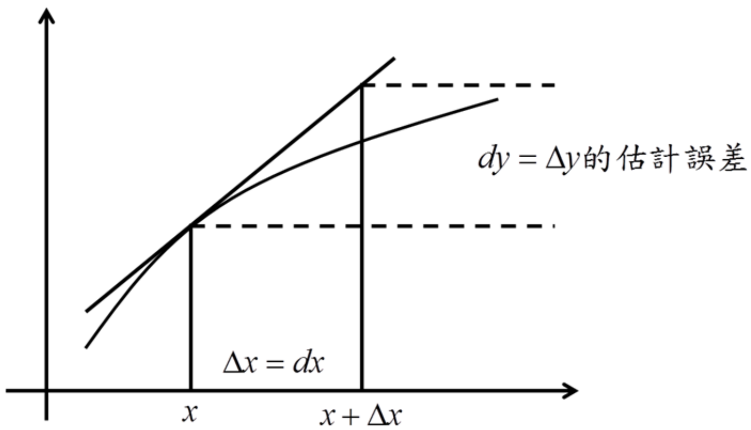
(ii). $f_x$, $f_y$ continuous at ( $a$, $b$ ).

**Remark**：Differentiable ⇒ Continuous.

＊Differentiable

Recall：

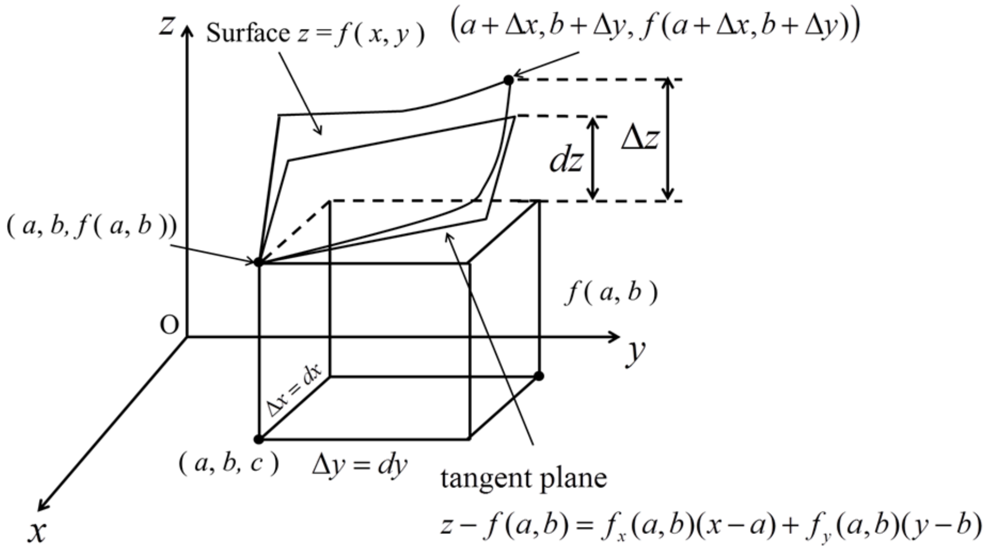
- 1 – D：




$\Delta y = f(x + \Delta x) - f(x) =$ 實際誤差

$\Delta y \approx dy = f'(x)dx$(切線代曲線得到的誤差)

- 2 – D：




$z = f(a + \Delta x, b + \Delta y) - f(a,b) =$ 實際誤差

$dz =$ 估計誤差(利用切平面代曲面得到的誤差)

$= f_x(a,b)dx + f_y(a,b)dy$

公式：$\Delta z \approx dz = f_x(a,b) + f_y(a,b)dy.$

**Example 4**：

$z = f(x, y) = x^2 + 3xy - y^2$

(i). Find $dz$.

(ii). Use differentials to estimate $f(2.05, 2.96)$.

**Solution**：

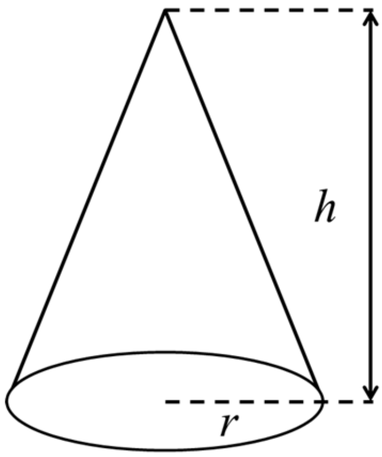
(i). $dz = (2x + 3y)dx + (3x - 2y)dy$

(ii). $dz = (2.2 + 3.3)(0.05) + (3.2 - 2.3)(-0.04) = 0.65$

(令$a = 2, b = 2 \Rightarrow \Delta x = 0.05, \Delta y = -0.04$).

＊註 $\Delta z = f(2.05, 2.96) - f(2,3) = 0.6449$.

**Example 5**：




$r = 10 \pm 0.1$

$h = 25 \pm 0.1$

Use differentials to estimate the maximum error in the calculated volume of the cone.

**Solution**：

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{2}{3}\pi h r dr + \frac{1}{3}\pi r^2 dh$$

$$= 20\pi$$