§14.3 Partial Derivatives

＊Partial Derivative (偏導數) $f_x(a, b)$ of f with respect to x at (a, b).

Definition：

(i). $f_x(a,b)=\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}$

(將 $y=b$ 固定，把 f 看成單變數 x 的微分)

(ii). 同理可定義對 y 的偏導數作用在(a, b)為

$f_y(a,b)=\lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}$

(將 $x=a$ 固定，把 f 看成單變數 y 的微分)

＊簡單來說，偏微就是將其他變量固定，對唯一變量作微分。因此偏微和上學期的單變數函數微分是一樣的。

＊按*讚*的問題：如何定義多變量函數的"全"微分或(正常)的微分？

Notation：

$$\frac{\partial f}{\partial x}=f_x=f_1 : \frac{\partial f}{\partial y}=f_y=f_2.$$

Example 1：

Let $f(x,y)=x^2+xy+y^2$, find $f_x, f_y, f_{xx}, f_{yy}, f_{xy}, f_{yx}$.

Solution：

$f_x=2x+y$ (把y當常數對x微分)

$f_y=2y+x$ (把x當常數對y微分)

$$f_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{11}=2$$

$$\left(f_x\right)_y=f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=f_{12}=1$$

$$\left(f_y\right)_x=f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=f_{21}=1$$

$$f_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{22}=2$$

Example 2：

$f(x,y,z) = xyz$, $x > 0$, find $f_z(2,3,0)$.

Example 3：

$$f(x,y)=\begin{cases}\dfrac{2x^2+3y^2(x^4+e^y)}{x-y} & , x\neq y \\ 0 & , x=y\end{cases} \quad \text{Find } f_x(0,0)$$

Theorem：

(i). (Clairau's Theorem)

If f_{xy} and f_{yx} are continuous on a region D, then

$$f_{xy}(a,b)=f_{yx}(a,b) \text{ for and } (a,b)\in D.$$

(ii). $f_{xyy}=\left(\left(f_x\right)_y\right)_y, f_{yxy}, f_{yyx}$ are continious on D, then

$$f_{xyy}(a,b)=f_{yxy}(a,b)=f_{yyx}(a,b) \text{ for } (a,b)\in D.$$

Example 2：

Let $u(x,y)=e^x\sin y$. Prove that $\Delta u=:\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

Proof：

$$u_x=e^x\sin y \qquad u_y=e^x\cos y$$
$$u_{xx}=e^x\sin y \quad u_{yy}=-e^x\sin y$$
$$\Rightarrow u_{xx}+u_{yy}=0 \ \text{(Laplace's Equation)}$$

Example 3：

Let $u(x,t)=f(x-at)$. Prove that $\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}.$

Proof：

$$u_t=-af'(x-at) \qquad u_x=f'(x-at)$$
$$u_{tt}=a^2f''(x-at) \quad u_{xx}=f''(x-at)$$
$$\Rightarrow t_{tt}=a^2u_{xx} \ \text{(Wave Equation)}$$

Example 4：

Let $xyz=\cos(x+y+z)$. Find $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}.$

Solution：

將 z 想成以 x 和 y 為獨立變數的函數

(Take partial derivative with respect to x)

$$\Rightarrow yz + xy\frac{\partial z}{\partial x} = \left(-\sin(x+y+z)\right)\left(\frac{\partial z}{\partial x}+1\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-\sin(x+y+z)-yz}{xy+\sin(x+y+z)}$$

Similarly,

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-\sin(x+y+z)-xz}{xy+\sin(x+y+z)}$$

Example 5：

$$f(x,y) = \left(x^3+y^3\right)^{\frac{1}{3}} e^{-\sin y}. \text{ Find } f_x(1,0).$$

Solution：

$$f(x,0) = \sqrt[3]{x^3} = x \Rightarrow f'(x) = 1$$

$$\Rightarrow f_x(1,0) = 1.$$

($f_x(1,0)$ 將y固定為0對x的微分.)

Example 6：

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2}, & (x,y) \neq (0,0). \\ 0 & , (x,y) = (0,0). \end{cases}$$

(1) Find $f_x(x,y)$, $f_y(x,y)$ for $(x,y) \neq (0,0)$.

(2) Find $f_x(0,0)$, $f_y(0,0)$.

(3) Find $f_{xy}(0,0)$, $f_{yx}(0,0)$.

Solution：

(1) For $(x,y) \neq (0,0)$, we have

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{\left(x^2+y^2\right)^2},\ f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{\left(x^2+y^2\right)^2}.$$

(2) By definition, we have

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = 0$$

(3)

Let

$$g(x,y) = f_x(x,y) = \begin{cases} \dfrac{x^4y + 4x^2y^3 - y^5}{\left(x^2+y^2\right)^2}, & (x,y) \neq (0,0). \\ 0, & (x,y) = (0,0). \end{cases}$$

$$\Rightarrow g_y(0,0) = f_{xy}(0,0) = \lim_{h\to 0} \frac{g(0,h) - g(0,0)}{h} = \frac{\dfrac{-h^5}{h^4}}{h} = -1.$$

Let

$$h(x,y) = f_y(x,y) = \begin{cases} \dfrac{x^5 - 4x^3y^2 - xy^4}{\left(x^2+y^2\right)^2}, & (x,y) \neq (0,0). \\ 0, & (x,y) = (0,0). \end{cases}$$

$$\Rightarrow h_x(0,0) = f_{yx}(0,0) = \lim_{l\to 0} \frac{h(0,l) - g(0,0)}{l} = \frac{\dfrac{l^5}{l^4}}{l} = 1.$$

Since $f_{xy}(0,0) \neq f_{yx}(0,0)$, by Clairaut's Theorem

f_{xy} and f_{yx} are not continuous near (0,0).

Check：

If $(x,y) \neq (0,0)$, then $f_{xy}(x,y)$ and $f_{yx}(x,y)$分母、分子皆為次數 4 的齊次項。

$\Rightarrow \lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ and $\lim_{(x,y)\to(0,0)} f_{yx}(x,y)$ do not exist.