§13.3 Arc Length and Curvature

1. **Arc length**： $r(t)=\langle x(t),y(t)\rangle$ or $\langle x(t),y(t),z(t)\rangle$

$$L=\int_a^b\sqrt{(dx)^2+(dy)^2}=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_a^b|r'(t)|dt$$

$$L=\int_a^b\sqrt{(dx)^2+(dy)^2+(dz)^2}=\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt=\int_a^b|r'(t)|dt$$

(i). 從物理角度看：

$r'(t)$代表物體在 t 時間的瞬間速度

$|r'(t)|$代表物體在 t 時間的瞬間速率

$|r'(t)|\Delta t=$ 小範圍的距離

$\Rightarrow \int_a^b|r'(t)|dt=$ 代表此物體從$t=a$到$t=b$所走過的距離

2. **Arc length function** (or Distance Function)

$s(t)=\int_a^t|r'(u)|du.$

$\Rightarrow \dfrac{ds}{dt}=|r'(t)|=$ 距離的變化率 $=$ 速度.

Example 1：

Find the length of the curve $r(t)=\left\langle \sin 2t,\cos 2t,2t^{\frac{3}{2}}\right\rangle, 0\le t\le 1.$

(A) $\frac{2}{27}\left(13\sqrt{13}-8\right)$ (B) $\frac{13}{9}$ (C) $\frac{13\sqrt{13}-6}{27}$ (D) $\frac{16}{9}$.

Solution：

(A)

Example 2：

Let C be a curve described by $x=f(t)$, $y=g(t)$, $\alpha\leqq t\leqq\beta$, where f' and g' are continuous on $[\alpha,\beta]$ and C is traversed exactly once as t runs from α to β. Which one of the following is always true?

(A) $\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \ge \beta-\alpha$

(B) $\int_{\alpha}^{\beta}\sqrt{\left|\frac{dx}{dt}\right|+\left|\frac{dy}{dt}\right|}\,dt \ge \sqrt{\beta^2+\alpha^2}$

(C) $\int_{\alpha}^{\beta}\sqrt{\left|\frac{dx}{dt}\right|+\left|\frac{dy}{dt}\right|}\,dt \ge \beta-\alpha$

(D) $\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \ge \sqrt{(y(\beta)-y(\alpha))^2+(x(\beta)-x(\alpha))^2}$

Solution：

(D)

Example 3：

Let the distance traveled by a particle with position

$(x(t), y(t)) = (\sin^2 t, \cos^2 t)$ as t varies from $t=0$ to $t=3\pi$ be d.

Then $d=$?

(A) $\sqrt{2}$ (B) 0 (C) $6\sqrt{2}$ (D) $4\sqrt{2}$

Solution：

(C)

3. 將一個 curve 用 arc length(s) 來作參數式是一個非常有用的想法和技巧.
(如此的表達方式不隨著不同座標系統而改變)

4. Unit tangent vector：$T(t)=\dfrac{r'(t)}{|r'(t)|}$.

5. **Curvature**(曲率)：a measure of how quickly the curve changes direction at a given point.

Definition：

$$\kappa = \text{曲率} = \left|\frac{dT}{ds}\right|$$

Example 4：

Show that the curvature of a circle with radius a is $\frac{1}{a}$.

Theorem：

(i). $\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT/dt}{ds/dt}\right| = \left|\frac{T'(t)}{r'(t)}\right| = \frac{|r'(t)\times r''(t)|}{|r'(t)|^3}$

(ii). Given a plane curve $y = f(x)$, then its curvature κ at a given point x is

$$\kappa(x) = \frac{|f''(x)|}{\left(1+\left(f'(x)^2\right)\right)^{3/2}}$$

Proof：

(i). $r' = |r'|\,T = \frac{ds}{dt}T$ (1)

$\Rightarrow r'' = \frac{d^2s}{dt^2}T + \frac{ds}{dt}\mathrm{T}'$ (2)

$(1)\times(2)$

$\Rightarrow r'\times r'' = \left(\frac{ds}{dt}\right)^2 (T\times T')$.

$\Rightarrow |r'\times r''| = \left(\frac{ds}{dt}\right)^2 |T||T'| = \left(\frac{ds}{dt}\right)^2 |T'|$

$\Rightarrow |T'| = \frac{|r'\times r''|}{\left(ds/dt\right)^2} = \frac{|r'\times r''|}{|r'|^2}$

$\Rightarrow \kappa = \frac{|T'|}{|r'|} = \frac{|r'\times r''|}{|r'|^3}$

6. Principal unit normal vector $N(t)$.

$$N(t) = \frac{T'(t)}{|T'(t)|}.$$

7. Binormal vector $B\ (t)$.

$B(t) = T(t) \times N(t)$