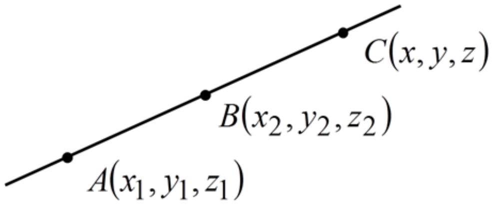
§12.5 Equations of Lines and Planes

∗**Equation of lines**：Given A and B, find the equation of the line $\overline{AB}$.



$\overrightarrow{AB}. =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >, \overrightarrow{AC} =< x - x_1, y - y_1, z - z_1 >$
$:=< a, b, c >$

$\overrightarrow{AB} // \overrightarrow{AC} \Rightarrow \exists t \in R$ s.t. $\overrightarrow{AC} = t\overrightarrow{AB}.$

(i). Parametric equation

$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}, \ t \in R.$$

(ii). Symmetric equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \ abc \neq 0$$

(If $a = 0, bc \neq 0 \Rightarrow x = x_1, \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$)

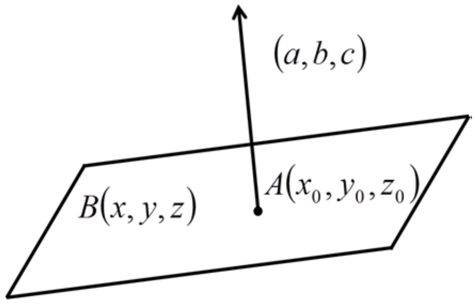
Remark：

$< a, b, c >$ is called the direction of the line $\overline{AB}$.

∗Equation of the plane：

Given the normal direction of the plane < a, b, c >

and a point (x_0, y_0, z_0) on the plane.



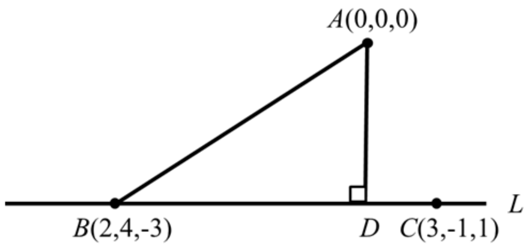
⇒ the equation of the plane：

i. $\langle a,b,c\rangle\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$

ii. $ax+by+cz = ax_0+by_0+cz_0 (=d)$

Example 1：

Line L though (2,4,-3), (3,-1,1). Find the distance between L and (0,0,0).



Solution：

$\overline{BA} = \langle -2,-4,3\rangle$ and $\overline{BC} = \langle 1,-5,-4\rangle$

Vector projection of $\overline{BA}$ on $\overline{BC}$.

$$= \overline{BD} = \left(\frac{\overline{BA}\cdot\overline{BC}}{\left|\overline{BC}\right|}\right)\cdot\frac{\overline{BC}}{\left|\overline{BC}\right|} = \left\langle \frac{1}{7}, \frac{-5}{7}, \frac{-4}{7}\right\rangle$$

$$\overline{AD} = \overline{AB} + \overline{BD} = \langle 2,4,-3\rangle + \left\langle \frac{1}{7}, \frac{-5}{7}, \frac{-4}{7}\right\rangle$$

$$= \left\langle \frac{15}{7}, \frac{23}{7}, \frac{-25}{7}\right\rangle$$

The distance $= \left|\overline{AD}\right|$.

Example 2：

Find the equation of the plane through (1,3,2), (3,-1,6) and (5,2,0).

Solution：

$$
\begin{aligned}
法向量 &= \langle 2,-4,4\rangle \times \langle 4,-1,-2\rangle \\
&= \langle 12,20,14\rangle \\
&= 2\langle 6,10,7\rangle
\end{aligned}
$$

$$\langle 6,10,7\rangle \cdot \langle x-5, y-2, z-0\rangle = 0$$

$$\Rightarrow 6x+10y+7z=50.$$

Example 3：

i. Find the angles between two planes P_1： $x + y + z = 1$ and P_2： $x - 2y + 3z = 1$.

ii. Let $l = P_1 \cap P_2$. Find the equation of l.

Solution：

i. 平面夾角 ＝ 法向量夾角

$$\Rightarrow \cos\theta = \frac{\langle 1,1,1\rangle \cdot \langle 1,-2,3\rangle}{\sqrt{3}\sqrt{14}} = \frac{2}{\sqrt{42}} = \frac{\sqrt{42}}{21}$$

$$\Rightarrow \theta = \cos^{-1}\frac{\sqrt{42}}{21}.$$

ii. l 的方向和 P_1，P_2 平面的法向量垂直：

$$\Rightarrow \langle 1,1,1\rangle \times \langle 1,-2,3\rangle = \langle 5,-2,-3\rangle = \text{直線}L\text{的方向。}$$

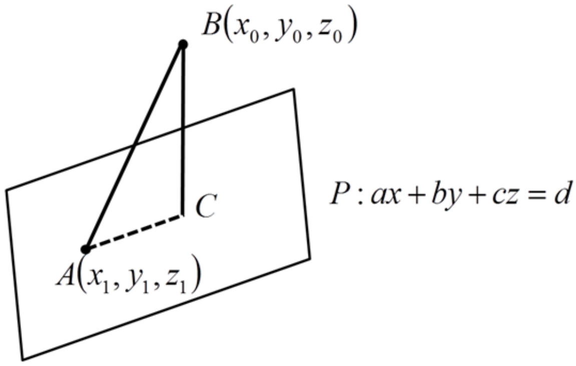
$$\Rightarrow \begin{cases} x+y+z=1 \\ x-2y+3z=1 \end{cases} \Rightarrow \text{Let } z=0 \Rightarrow y=0, x=1.$$

$$\Rightarrow \frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}.$$

Example 4：

Find the distance between two parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Solution：



$$\overline{BC} = \left|\text{Scalar projection of } \overline{BA} \text{ on } \overline{BC}\right|.$$

$$= \left|\frac{\overline{BA} \cdot \overline{BC}}{\overline{BC}}\right|$$

$$= \left|\frac{ax_0 + by_0 + cz_0 - d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

$=$ 點到平面之距離.

$$\left(\overline{BA} = (x_1 - x_0, y_1 - y_0, z_1 - z_0),\ \overline{BC} = (a, b, c)\right)$$

Now, let $x = 0, y = \dfrac{5}{2}, z = 0.$

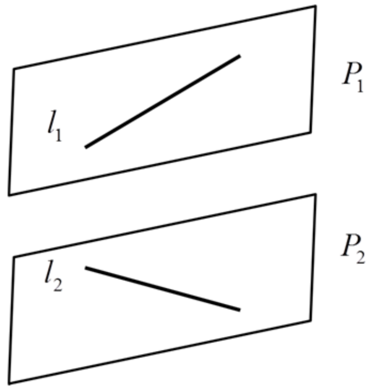
$\Rightarrow$ 平行平面之距離 $= \dfrac{\frac{5}{2}}{\sqrt{27}} = \dfrac{5\sqrt{3}}{18}.$

Example 5：

Find the distance between two skew lines

$$l_1: \begin{cases} x = 1 + t \\ y = -2 + 3t, \\ z = 4 - t \end{cases} t \in R,\ l_2: \begin{cases} x = 2s \\ y = 3 + s \\ z = -3 + 4s \end{cases},\ s \in R.$$

Solution：



Find $P_1 // P_2$ and $l_1 \in P_1, l_2 \in P_2$.

$\Rightarrow$ The normal line of P_1 and P_2 $n \perp l_1, l_2$.

$\Rightarrow n =< 1,3,1 > \times < 1,1,4 >=< 11. -3. -2 >$

Find the distance between P_1 and P_2.

$\Rightarrow d(l_1, l_2) = d(P_1, P_2)$.