§12.4 The Cross Product

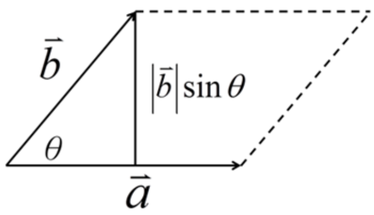
∗Cross Product (a vector) $\vec{a}\times\vec{b}$

Let $\vec{a}=<a_1,a_2,a_3>$, $\vec{b}=<b_1,b_2,b_3>$.

1. 代數意義：$\vec{a}\times\vec{b}=\left(\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix},\begin{vmatrix}a_3 & a_1\\ b_3 & b_1\end{vmatrix},\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\right)=\begin{vmatrix}i & j & k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

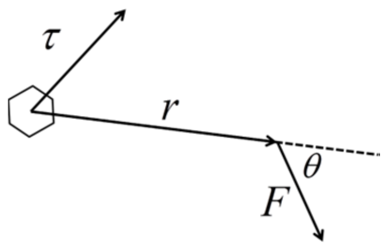
   $i=<1,0,0>$ $j=<0,1,0>$ $k=<0,0,1>$

2. 幾何意義：$\vec{a}\times\vec{b}=\begin{cases}\text{大小：}|a||b|\sin\theta\\ \text{方向：right hand rule.}\end{cases}$



   $\Rightarrow\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta=$ the parallelogram determined by $\vec{a}$ and $\vec{b}$.

3. 物理意義：a force F acting on a rigid body at a point given by a position vector r. Then the torque τ is defined to be $\tau=r\times F$.

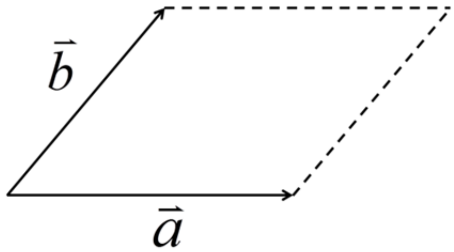


4. $\vec{a}\times\vec{b}=0\Leftrightarrow\vec{a}\,//\,\vec{b}$.

5. $\left(\vec{a}\times\vec{b}\right)\perp\vec{a}$, $\left(\vec{a}\times\vec{b}\right)\perp\vec{b}$.

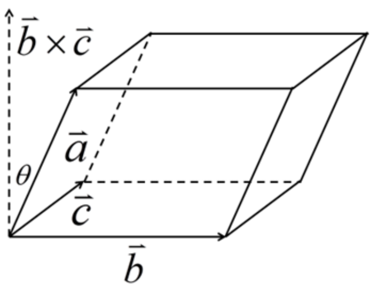
6.

- Area of the parallelogram bounded by $\vec{a}$ and $\vec{b} = |\vec{a}||\vec{b}|\sin\theta = |\vec{a}\times\vec{b}|$.



- Volume of the parallelepiped bounded by $\vec{a},\vec{b},\vec{c}$

$$|\vec{a}\cdot(\vec{b}\times\vec{c})| = \underbrace{|\vec{a}||\cos\theta|}_{\text{高}} \times \underbrace{|\vec{b}\times\vec{c}|}_{\text{底面積}}$$



7.

i. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}.$

ii. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}.$

iii. $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}.$
(×和·可交換，由$\vec{a},\vec{b},\vec{c}$張出的體積來理解)

iv. $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} \neq (\vec{a}\times\vec{b})\times\vec{c}.$
(×不可結合)

Remark：

1. $i\times(i\times j) = i\times k = -j.$
$(i\times i)\times j = 0\times j = 0.$

2. Why $\vec{a}\times(\vec{b}\times\vec{c})$在$\vec{b}$ & $\vec{c}$ 所張的平面？
$(\vec{a}\times\vec{b})\times\vec{c}$ 在哪兩個向量所張的平面？

Example 1：

Find the area of ΔPQR, where $P(1,4,6)$, $Q(-2,5,-1)$ and $R(1,-1,1)$.

Solution：

$\overrightarrow{PQ} = <-3,1,-7>,\ \overrightarrow{PR} = <0,-5,-5>$

$\overrightarrow{PQ} \times \overrightarrow{PR} = <-40,-15,15>$

$$\Delta PQR = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{5}{2}\sqrt{82}.$$

Example 2：

Are $\vec{a} = <1,4,-7>, \vec{b} = <2,-1,4>$ and $\vec{c} = <0,-9,18>$ coplanar(共面)?

Solution：

$<1,4,-7> \cdot (<2,-1,4> \times <0,-9,18>)$

$= <1,4,-7> \cdot <18,-36,-18> = 0$

$\Rightarrow$ Coplanar.