§11.9 Representations of Functions as Power Series

- 源頭函數：$\dfrac{1}{1-x}=1+x+x^2+\ldots\ldots=\sum_{n=0}^{\infty}x^n,\ |x|<1.$

(i). 直系函數：

Example 1：

$$\frac{1}{1+x^2}=\frac{1}{1-\left(-x^2\right)}=\sum_{n=0}^{\infty}\left(-x^2\right)^n=\sum_{n=0}^{\infty}(-1)^n x^{2n},\ \left|x^2\right|<1\Leftrightarrow |x|<1.$$

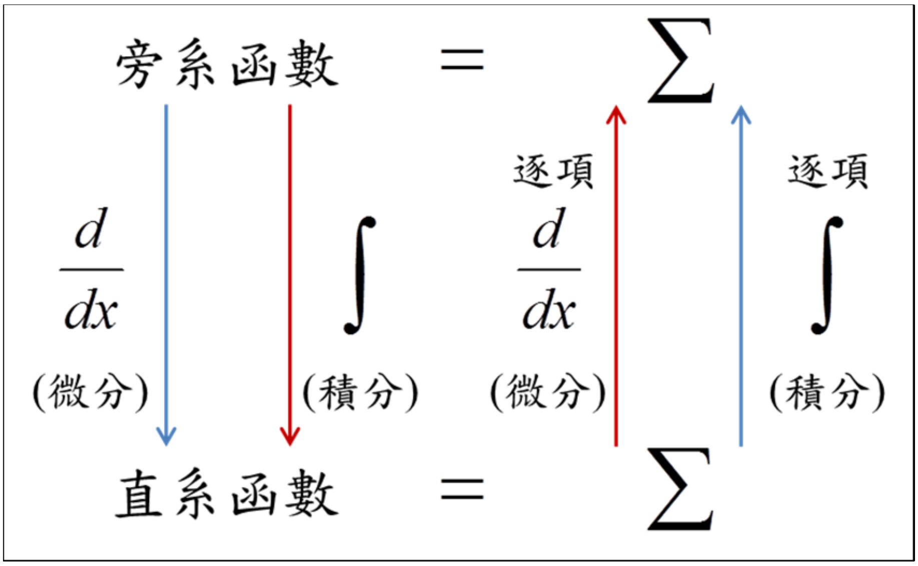
Example 2：

$$\frac{1}{2-x}=\frac{1}{2}\frac{1}{1-\frac{x}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}},\ |x|<2.$$

Example 3：

$$\frac{x^3}{x+2}=\frac{x^3}{2}\frac{1}{1-\left(-\frac{x}{2}\right)}=\frac{x^3}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+3}}{2^{n+1}},\ |x|<2.$$

(ii). 旁系函數：(related to 直系函數 by 微分和積分)



＊註記：

1. Power series 在絕對收斂的範圍(即收斂區間)可作逐項微分或積分的動作。
2. 有時須微分或積分不只一次才到直系。

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Example 4：

(i). Find the power series representation of ln (1- x).

(ii). Express ln 2 as a convergent infinite series with all positive terms.

Solution：

$$\underbrace{\ln(1-x)}_{\downarrow \frac{d}{dx}} = \underbrace{c - x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots}_{\uparrow \int} = c - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

$$\frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n = -1 - x - x^2 - \ldots\ldots, \ |x| < 1.$$

Let $x = 0 \Rightarrow \ln 1 = 0 = c$.

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \ \ |x| < 1$$

$$= -\sum_{n=1}^{\infty} \frac{x^n}{n}, \ \ |x| < 1.$$

Example 5：

(i). Find the power series representation of $f(x) = \tan^{-1} x$.

(ii). Let $f(x) = \int_0^x \tan^{-1} t dt = \sum_{n=0}^{\infty} a_n x^n$. Then $a_8 = ?$

Solution：

$$\underbrace{\tan^{-1} x}_{\downarrow \frac{d}{dx}} = \underbrace{c + \sum_{n=0}^{\infty} (-1)^n x^{2n}}_{\uparrow \int} \ \ |x| < 1.$$

$$\frac{1}{1+x^2} = -\sum_{n=0}^{\infty} (-1)^n x^{2n}, \ |x| < 1.$$

Let $x = 0 \Rightarrow \tan^{-1} 0 = 0 = c + 0 \Rightarrow c = 0$.

$$\Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \ |x| < 1.$$

Example 6：

Find the power series representation of $f(x) = \dfrac{1}{(1-x)^2}$.

Solution：

$$\underbrace{\frac{1}{(1-x)^2}}_{\downarrow\int} = \underbrace{\sum_{n=1}^{\infty} n x^{n-1}}_{\uparrow\frac{d}{dx}}, \ |x|<1.$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \ |x|<1.$$

Example 7：

Evaluate $\int \frac{1}{1+x^7} dx$ as a power series.

Solution：

$$\int \frac{1}{1+x^7} dx$$

$$= \int \sum_{n=0}^{\infty} (-1)^n x^{7n} dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n x^{7n} dx$$

$$= \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} \right) + c \quad |x|<1.$$

Example 8：

1. $\sum_{n=1}^{\infty} n x^{n-1} = ?, \ |x|<1.$

2. (i) $\sum_{n=1}^{\infty} n x^n = ?, \ |x|<1.$

   (ii) $\sum_{n=1}^{\infty} \frac{n}{2^n} = ?$

3. (i) $\sum_{n=2}^{\infty} n(n-1)x^n = ?, \ |x|<1.$

   (ii) $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = ?$

   (iii) $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = ?$

Solution：

1. For $|x| < 1$, we have

$$\sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} \frac{d}{dx}\left(x^n\right) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

2. (i) For $|x|<1$, we have

$$\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$$ (此例子為旁系+直系)

(ii) $$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2.$$

3. (i) For $|x|<1$, we have $\sum_{n=2}^{\infty} n(n-1)x^n$

$$x^2\sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\sum_{n=2}^{\infty} \frac{d}{dx}\left(nx^{n-1}\right) = x^2\frac{d}{dx}\left(\sum_{n=2}^{\infty}\left(nx^{n-1}\right)\right) = \frac{2x^2}{(1-x)^3}.$$

(ii) $$\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \frac{2\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^3} = 4.$$

(iii) $$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=2}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4+2=6.$$