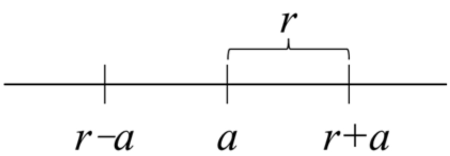
§11.8 Power Series

Definition： $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$ is called a power series about *a* (centered at *a*) (or in (*x* - *a*)).

Question：For what values of *x* do the corresponding series converge?



越靠近 a (中心)，越可能收斂。

- 收斂半徑(radius of convergence)
- 收斂區間(interval of convergence)

Example 1：

Find the interval of convergence and radius of convergence of $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$.

Solution：

Using ratio test, we get

$$\lim_{n\to\infty} \frac{\left|\frac{(x-3)^{n+1}}{n+1}\right|}{\left|\frac{(x-3)^n}{n}\right|} = \lim_{n\to\infty} \frac{n|x-3|}{n+1} = |x-3|.$$

If $|x-3| < 1$, that is $x \in (2,4)$, the power series converges (absolutely).

- Check end points：

$x = 2 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Rightarrow$ convergence.

$\text{x} = 4 \Rightarrow \sum_{\text{n}=1}^{\infty} \frac{1}{\text{n}} \Rightarrow$ divergence.

$\Rightarrow$ interval of convergence: $[2,4)$.

$\Rightarrow$ radius of convergence: $r = 1$.

Example 2：

Find the interval of convergence and radius of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Solution：

By ratio test

$$\Rightarrow \lim_{n\to\infty} \frac{\left|\dfrac{x^{2(n+1)}}{2^{2(n+1)}((n+1)!)^2}\right|}{\left|\dfrac{x^{2n}}{2^{2n}(n!)^2}\right|}$$

$$\Rightarrow \lim_{n\to\infty} \frac{x^2}{4(n+1)^2} = 0 \ \forall x \in R$$

$\Rightarrow$ interval of convergence : $(-\infty, \infty)$

$\Rightarrow$ radius of convergence : $r = \infty$

＊註：重量級在分母 $(n!)^2 >> x^{2n}$.

Example 3：

Find the values of x for which the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}3^n}(x-1)^n$ converges

Solution：

Example 4：

Find the interval of convergence and radius of convergence of $\sum_{n=1}^{\infty} n!x^n$

Solution：

By ratio test

$\Rightarrow \lim_{n\to\infty}(n+1)$ (not exist $\forall x \in R$)

$\Rightarrow$ interval of convergence : $\{0\}$.

$\Rightarrow$ radius of convergence : r = 0.

＊註：重量級$(n! >> x^n)$在分子。

Example 5：

Find the interval of convergence and radius of convergence of

$$\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n}.$$

Solution：

By ratio test

$$\Rightarrow \lim_{n\to\infty} \left| \frac{2(x-1)n}{n+1} \right| = 2|x-1| \ \forall x \in R.$$

$$\text{Let } 2|x-1| < 1 \Rightarrow |x-1| < \frac{1}{2} \Rightarrow x \in \left(\frac{1}{2}, \frac{3}{2} \right).$$

- Check end points：

$$x = \frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Rightarrow \text{convergence.}$$

$$x = \frac{3}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \quad \Rightarrow \text{divergence.}$$

$$\Rightarrow \text{interval of convergence} : x \in \left[\frac{1}{2}, \frac{3}{2} \right).$$

$$\text{radius of convergence} : r = \frac{1}{2}.$$

Example 6：

Find a value of b that will make the radius of convergence of power series $\sum_{n=2}^{\infty} \frac{b^n x^n}{\ln n}$ equal to 5.

Solution：

$$a_n = \frac{b^n x^n}{\ln n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{b^{n+1} x^{n+1}}{\ln(n+1)} \times \frac{\ln n}{b^n x^n} \right| = |bx| \lim_{n\to\infty} \left| \frac{\ln n}{\ln(n+1)} \right| = |bx|$$

$$\text{By the Ratio Test, Let } |bx| < 1 \Rightarrow -1 < bx < 1 \Rightarrow \frac{-1}{b} < x < \frac{1}{b}$$

$$\Rightarrow \because r = 5 \therefore b = \frac{1}{5}.$$

Example 7：

For which α is the interval of convergence of $\sum_{n=1}^{\infty} \frac{n^\alpha}{n+1} x^n$ equal to $[-1,1)$

Solution：

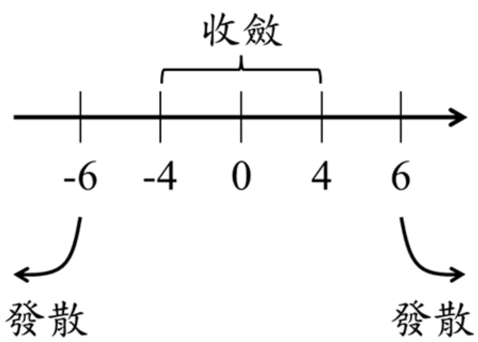
$$\sum_{n=1}^{\infty} \frac{n^\alpha}{n+1} x^n$$

Example 8：

If $\sum_{n=1}^{\infty} c_n x^n$ converges with $x = 4$ and diverges when $x = 6$, then determine the convergence of the following series.

Solution：

		answers	reasons
i.	$\sum_{n=1}^{\infty} c_n$	convergence	$x = 1$
ii.	$\sum_{n=1}^{\infty} c_n 8^n$	divergence	$x = 8$
iii.	$\sum_{n=1}^{\infty} c_n (-3)^n$	convergence	$x = -3$
iv.	$\sum_{n=1}^{\infty} (-1)^n c_n 9^n$	divergence	$x = 9$



＊註： If $4 \leq |x| \leq 6$, then can't determine whether it is convergent ot divergent.

Example 9：

The power series $\sum_{n=1}^{\infty} a_n (x-2)^n$ and $\sum_{n=1}^{\infty} b_n (x-3)^n$ both converge at $x = 6$.

Find the largest interval over which both series must converge.

Solution：

Theorem：

For series $\sum_{n=1}^{\infty} c_n (x-a)^n$, either one of the following 3 possibilities holds：

(i). $r = 0$：只有一點收斂

(ii). $r < \infty$ ：兩端點須被檢驗才可知其收斂與否

(iii). $r = \infty$ ：每一點接收斂