§11.6 Absolute Convergence and The Ratio and Root test

Definition：

i. $\sum_{n=1}^{\infty} a_n$ is said to be **absolute convergence** (AC)

if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

ii. If $\sum_{n=1}^{\infty} a_n$ is convergent but not AC,

then $\sum_{n=1}^{\infty} a_n$ is called **conditional convergence**(CC).

Example 1：

i. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ ：CC.

ii. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ ：AC.

Theorem 1：

AC ⇒ convergence (反之不對).

＊Ratio and Root Tests(for AC)

- **Who**：Anyone.
- **What**：這兩個方法的精神都是在檢查尾巴項的行為。

  且都以等比級數$\sum_{n=1}^{\infty} a_n r^n$ 作為比較的對象。

i. Ratio Test：比前後項縮小的比例

- $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ (若為等比級數，L即為公比)
- 適用對象：有“！”項.

ii. Root Test：開 n 次根號也為檢查尾巴項行為的方法之一

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

  (若為等比級數，則$\sqrt[n]{|a_0| r^n} = |a_0|^{\frac{1}{n}} r = r$，即$L$為公比)

- 適用對象：有 $\sqrt[n]{\ }$ or $(\ )^n$ 項.

＊結論：

i. $L<1 \Rightarrow$ AC

ii. $L>1 \Rightarrow$ Divergence

> ＊註記：
>
> 1. $n^n >> n! >> e^n >> n^2 >> \ln n$. 重量級在分母，必 AC.
>
> 2. P-series 用此 2 tests 都檢查不出來。

iii. $L=1 \Rightarrow$ The test fails. $\left(\sum_{n=1}^{\infty}\frac{1}{n} \quad \sum_{n=1}^{\infty}\frac{1}{n^2}\right)$

(P-series 用此 2 tests 都檢查不出來。)

Example 2：

		Test	AC, CC or Div.
i.	$\sum_{n=1}^{\infty}\frac{n^2}{2^n}$	$\sqrt{\ }$, $\frac{n^{\frac{2}{n}}}{2}=\frac{1}{2}$	AC
ii.	$\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$	$\sqrt{\ }$, $\frac{1}{2}$	AC
iii.	$\sum_{n=1}^{\infty}\frac{(-3)^n}{n!}$	$-$, $\left\lvert\frac{3}{n+1}\right\rvert \to 0$	AC
iv.	$\sum_{n=1}^{\infty}(-1)^n\frac{2^n n!}{5\times 8\times \ldots \times (3n+2)}$	$-$, $\left\lvert\frac{2(n+1)}{3(n+1)+2}\right\rvert \to \frac{2}{3}$	AC
v.	$\sum_{n=1}^{\infty}(-1)^n\frac{5^n n!}{5\times 8\times \ldots \times (3n+2)}$	$-$, $\left\lvert\frac{5(n+1)}{3(n+1)+2}\right\rvert \to \frac{5}{3}$	Div.

vi. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(\tan^{-1} n\right)^n}$ $\quad\sqrt{\ }$, $\left|\frac{1}{\tan^{-1} n}\right| \to \frac{2}{\pi}$ AC

vii. $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}}$ $\quad \approx \sum_{n=1}^{\infty} \frac{\frac{\pi}{2}}{n^{\frac{3}{2}}}$ AC

as n is large

＊註：p-級數不需用(也不能用)此 2 tests.

Example 3：

For which positive integers k is the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ convergent?

Solution：

$$\frac{\frac{((n+1)!)^2}{(k(n+1))!}}{\frac{(n!)^2}{(kn)!}} = \frac{(n+1)^2}{(kn+1)(kn+2)...(kn+k)}$$

$\Rightarrow$ When $k \geq 2, L < 1.$ (convergent)

＊補充：

Sterling Formula

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} < n! < \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1+\frac{1}{4n}\right).$$

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Example 1：

Determine the convergence / divergence of the series $\sum_{n=1}^{\infty} \frac{4^n (n!)}{(2n)!} = \sum_{n=1}^{\infty} a_n$.

Solution：

By using the Sterling formula, we have

$$a_n = \frac{(n!)}{(2n)!} 4^n \sim \frac{\left[\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right]^2}{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi(2n)}} = \frac{2\pi n}{\sqrt{4\pi n}} \to \infty \text{ as } n \to \infty$$

$\Rightarrow$ The series diverges.