§11.4 The Comparison Tests

- **Who**：$\sum_{n=1}^{\infty} a_n,\ a_n > 0,\ \forall n$(或從某項以後恆正/恆負)

  (即對象為 essentially positive or negative series.)

- **What**：

  1. 減法比較：

     (i). $a_n \geq b_n$ and $\sum a_n$ converges $\Rightarrow \sum b_n$ converges.

     (ii). $a_n \geq b_n$ and $\sum b_n$ diverges $\Rightarrow \sum a_n$ diverges.

  2. 除法比較：

     (i). $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$(尾巴的速度相同)

     $\Rightarrow$ Both $\sum a_n$ and $\sum b_n$ converge or diverge.

     (ii). $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$(a_n 跑到 0 的速度比較快)

     $$\begin{cases} \text{If } \sum b_n \text{ converges} \Rightarrow \sum a_n \text{ converges.} \\ \text{If } \sum a_n \text{ diverges} \Rightarrow \sum b_n \text{ diverges.} \end{cases}$$

＊註記：若 $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$(b_n 跑到 0 速度比 a_n 快)，可得類似結論。

Example 1：

i.	$\sum_{n=1}^{\infty}\frac{1}{n^2-n-1}$	$\sum_{n=1}^{\infty}\frac{1}{n^2}$	Conv.
ii.	$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)(n+2)}}$	$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$	Conv.
iii.	$\sum_{n=1}^{\infty}\frac{n^3}{2^n}$	$\sum_{n=1}^{\infty}\frac{1}{2^n}$	Conv.
iv.	$\sum_{n=1}^{\infty}\sin\frac{1}{n}$	$\sum_{n=1}^{\infty}\frac{1}{n}$	Div.
v.	$\sum_{n=1}^{\infty}\frac{n+5}{\sqrt[3]{n^7+n^2}}$	$\sum_{n=1}^{\infty}\frac{1}{n^{\frac{4}{3}}}$	Conv.
vi.	$\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{1}{n}}}$	$\sum_{n=1}^{\infty}\frac{1}{n}$	Div.
vii.	$\sum_{n=1}^{\infty}\frac{\ln n}{\sqrt{ne^n}}$	$\sum_{n=1}^{\infty}\frac{1}{e^n}$	Conv
viii.	$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{1+n^2}\right)$		Conv.

Example 2：

If $\sum a_n$ converges, where $a_n>0$, is it true that $\sum\sin a_n$ is also convergent?

Solution：

Yes.

Since $a_n\to 0$ as $n\to\infty$.

And $\frac{\sin a_n}{a_n}\to 1$ as $n\to\infty$.

$\Rightarrow \sum\sin a_n$ converges.

Example 3：

If $\sum a_n$, $\sum b_n$ are both convergent series with positive terms, is it true that $\sum a_n b_n$ is also convergent?

Solution：

Yes.

For $a_n \to 0$ and $b_n \to 0$.

and $a_n b_n$ approaches 0 even faster than a_n (or b_n).

$\Rightarrow \sum a_n b_n$ converges.