§11.3 The Integral Test and Estimates of Sums

- **Who**：What kind of series can apple the integral test to check its convergence?

a_i： (i) essentially positive (ii) decreasing (iii) $f(x) = a_x$ is continuous	(11.3-1)

⇓

- **What**：$\sum_{i=1}^{\infty} a_i$ and $\int_1^{\infty} a_x dx$ both converges or diverges.

Example 1：

Convergence or divergence? $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (p-series)

Recall：$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \text{convergence}, & p > 1 \\ \text{divergence}, & p \le 1 \end{cases}$, which implies the following theorem.

Theorem 1：

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergence}, & p > 1 \\ \text{divergence}, & p \le 1 \end{cases}$$

Example 2：

Convergence or divergence? $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution：

$$\int_1^{\infty} \frac{1}{x^2+1} dx = \tan^{-1} x \Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ (converge)}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ (converge)}$$

Example 3：

Convergence or divergence? $\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Solution：

$$\int_1^\infty \frac{\ln x}{x}\,dx = \int_0^\infty u\,du \quad (u = \ln x).$$

$\Rightarrow$ diverges

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\ln n}{n} \text{ diverges}$$

Example 4：

If $\sum_{n=1}^{\infty}\left(\frac{a}{n+2}-\frac{1}{n+4}\right)$ is convergent where a is a constant, then a = ?

Solution：

a = 1.

Example 5：

Convergence or divergence? $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$.

Solution：

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \int_2^\infty \frac{du}{u^p} \quad (u = \ln x)$$

Example 6：

Convergence or divergence? $\sum_{n=1}^{\infty} b^{\ln n}$.

Solution：

$$\sum_{n=1}^{\infty} b^{\ln n} = \sum_{n=1}^{\infty}\left(e^{\ln b}\right)^{\ln n} = \sum_{n=1}^{\infty}\left(e^{\ln n}\right)^{\ln b} = \sum_{n=1}^{\infty} n^{\ln b}$$

$$\Rightarrow \begin{cases} \ln b \ge -1 \Rightarrow b \ge e^{-1} \Rightarrow \text{divergence.} \\ \ln b < -1 \Rightarrow b < e^{-1} \Rightarrow \text{convergence.} \end{cases}$$

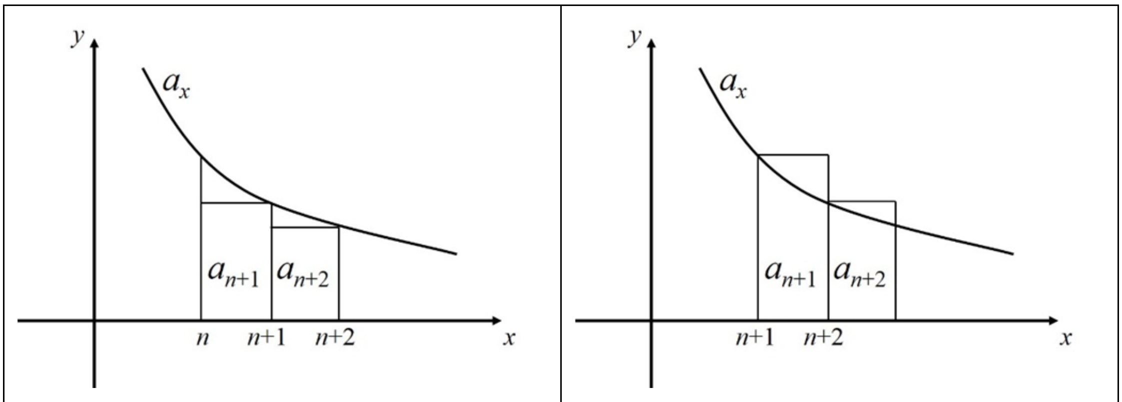
Theorem 2：

Let $\{a_i\}$ satisfy (11.3-1), and let $R_n = S - S_n$.

Here $S = \sum_{i=1}^{\infty} a_i$ and $S_n = \sum_{i=1}^{n} a_i$.

Then $\int_{n+1}^{\infty} a_x\,dx \le R_n \le \int_n^\infty a_x\,dx$

Proof：



$$R_n = a_{n+1} + a_{n+2} + \ldots \le \int_n^{\infty} a_x\,dx \qquad R_n = a_{n+1} + a_{n+2} + \ldots \ge \int_{n+1}^{\infty} a_x\,dx$$

Example 7：

Use Theorem 2 to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ with $n = 10$.

Solution：

$$\int_{11}^{\infty} \frac{1}{x^3}\,dx \le S - S_{10} \le \int_{10}^{\infty} \frac{1}{x^3}\,dx$$

$$\Rightarrow \frac{1}{2(11)^2} \le S - S_{10} \le \frac{1}{2(10)^2}$$

$$\Rightarrow S_{10} + \frac{1}{2(11)^2} \le S \le S_{10} + \frac{1}{2(10)^2}$$

$$S_{10} \approx 1.197532$$

$$\Rightarrow 1.201664 \le S \le 1.202532$$

Example 8：

Let $t_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n.$

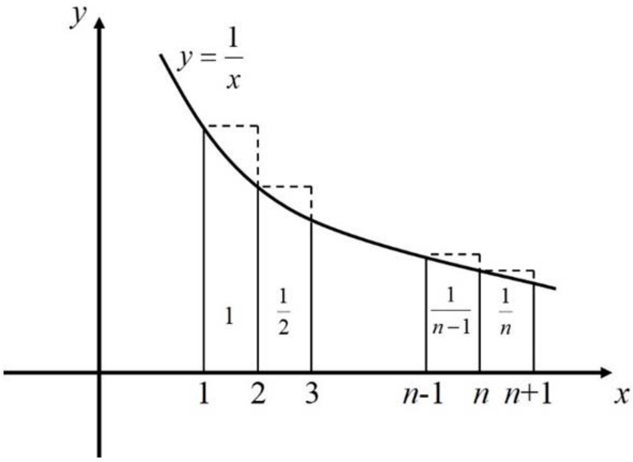
(a) Show that $t_n > 0.$

(b) Show that $t_n - t_{n+1} > 0.$

(c) Prove that $t_n \le 1$ for all $n.$

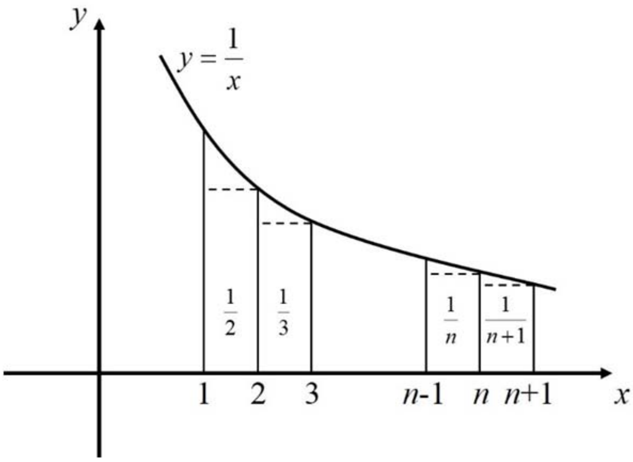
Solution：

(a)



由上圖得知 $t_n > 0$

(b)



由上圖得知 $\ln(n+1)-\ln(n) > \dfrac{1}{n+1} > 0$

$\Rightarrow t_n - t_{n+1} > 0$

$\Rightarrow t_n$ is a decreasing sequence.

(c)

由(b)得，$t_1 \geq t_n$ for all n. 但 $t_1 = 1$

$\Rightarrow 1 \geq t_n$ for all n.