§11.2 Series

Definition：

- Series： $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \ldots + a_n + \ldots$
- The convergence/divergence of a series：(n-th) partial sum $= S_n = \sum_{i=1}^{n} a_i$.

  $\{S_n\}$ converges/diverges $\Leftrightarrow$ $\sum_{i=1}^{\infty} a_i$ converges/diverges.

Theorem 1：(Geometric Series.)

$\sum_{i=1}^{\infty} r^i$ converges $\Leftrightarrow |r|<1$ and $\sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$.

Theorem 2：

$\sum_{i=1}^{\infty} a_i$ converges $\Rightarrow \lim_{n\to\infty} a_n = 0$.

$\left(\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.}\right)$

Remark：

$\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$$\because \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \underbrace{\left(\frac{1}{3}+\frac{1}{4}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{9}+\ldots+\frac{1}{16}\right)}_{>\frac{1}{2}} + \ldots\ldots \quad \text{(Diverges)}$$

Example 1：

Convergence or divergence?

(i). $\sum_{n=1}^{\infty} (-1)^n \Rightarrow$ diverges.

(ii). $\sum_{n=1}^{\infty} \frac{\ln n}{n} \Rightarrow$ diverges. $\left(\because \frac{\ln n}{n} > \frac{1}{n} \text{ when } n \geq 3\right)$

(iii). $\sum_{n=1}^{\infty} \frac{n}{n+1} \Rightarrow$ diverges. $\left(\because \frac{n}{n+1} \to 1 \text{ when } n \to \infty\right)$

Example 2：

Convergence or divergence? $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$.

Solution：

$$\sum_{n=1}^{\infty} 2^{2n} \times 3^{1-n} = \sum_{n=1}^{\infty} 4^{n} \times 3^{-n} \times 3 = \sum_{n=1}^{\infty} 3 \times \left(\frac{4}{3}\right)^{n}$$

$r = \frac{4}{3} > 1 \Rightarrow$ diverges.

Example 3：

Find the sum of the series $\sum_{n=1}^{\infty} (n-1) \times \left(\frac{1}{2}\right)^{n}$.

Example 4：

The Cantor Set, named after the German mathematician Georg Cantor(1845-1918), is constructed as follows. We start with the closed interval [0,1] and removed the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the two intervals $\left[0, \frac{1}{3}\right)$ and $\left(\frac{2}{3}, 1\right]$, and we remove the open middle third of each. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the proceeding step. The Cantor set consists of the number that remain in [0,1] after all those interval have been removed.

(i). Show that the total length of all the intervals that are removed is 1.

(ii). Give examples of some numbers in the Cantor Set.

Proof：

(i).

$$S = \frac{1}{3} + 2\left(\frac{1}{3}\right)^{2} + 2^{2}\left(\frac{1}{3}\right)^{3} + \ldots + 2^{2}\left(\frac{1}{3}\right)^{n+1} + \ldots$$

$$= \frac{1}{3}\left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^{2} + \ldots + \left(\frac{2}{3}\right)^{2} + \ldots\right)$$

$$= \frac{1}{3} \frac{1}{1 - \frac{2}{3}} = 1$$

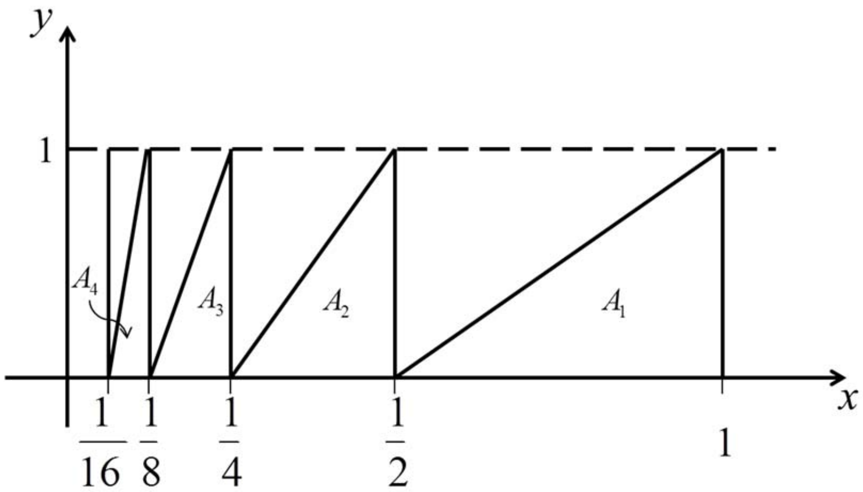
(ii). $\frac{1}{3},\frac{2}{3},\frac{1}{9},\frac{2}{9},\frac{7}{9},\frac{8}{9}\ldots$

Example 5：

$$\sum_{n=1}^{\infty} \ln \frac{\tan^{-1}(n+1)}{\tan^{-1}(n)} = ?$$

Example 6：

A sequence of right triangles A_1, A_2, A_3, … is given in the figure below



i. Let $a_n = Area\,(A_n)$. Determine an expression for a_n and find the limit of a_n.

ii. Let $b_n = \sum_{k=1}^{n} a_k$. Determine the limit of b_n.

Solution：

i. $\left\{a_n = \frac{1}{2^n}\right\}_{n=2}^{\infty}$ $a_n \to 0$.

ii. $\frac{1}{2}$.