§11.11 Applications of Taylor Polynomial

I. Approximating Function by Polynomials
(Note: Approximating Functions by sin *x* and cos *x* leads to the subject of Fourier expansion.)

II. Applications to Physics.

$$* \; f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Theorem：

$$|R_n(x)|=|f(x)-T_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}, \text{ where } |f^{(n+1)}(x)|\le M.$$

Example 1：

(i). What is the maximum error possible in using the approximation

$$\sin x\approx x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

when $-0.3\le x\le 0.3$? Use this approximation to find $\sin 12^\circ$ correct to six decimal places.

(ii). For what values of *x* this approximation accurate to within 0.00005?

Solution：

(i).

- $\left|\frac{x^7}{7!}\right|\le\frac{(0.3)^7}{7!}\approx 4.3\times 10^{-8}$

  (用這節的誤差估計或者交錯級數的誤差估計皆一樣)

- $\sin 12^\circ=\sin\frac{\pi}{15}\approx 0.20791169$

(ii). $\frac{|x|^7}{7!}<0.00005\Rightarrow |x|<(0.252)^{\frac{1}{7}}\approx 0.821.$

Example 2：

Einstein's theory of special relativity：

- $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$ $\quad \begin{cases} m_0 : \text{the mass of the object when at rest} \\ c : \text{the speed of light} = 3\times 10^8\ {}^{m}\!/\!_{s} \end{cases}$

- Kinetic energy : $K = mc^2 - m_0c^2$

- Classical kinetic energy : $K = \dfrac{1}{2}m_0v^2$

i. Show that when $v << c$, Kinetic energy≒Classical kinetic energy.

ii. Estimate its difference when $|v| \le 100\ {}^{m}\!/\!_{s}$.

Solution：

$$\text{i.}\quad (1+x)^{-\frac{1}{2}} = 1-\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3+\ldots$$

$$= 1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+\ldots$$

$$\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\frac{5}{16}\frac{v^6}{c^6}+\ldots$$

$$mc^2-m_0c^2 = m_0c^2\left[\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}-1\right]$$

$$= m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\ldots\right)$$

$$\approx \frac{1}{2}m_0v^2$$

$$\text{ii.}\quad f(x) = mc^2-m_0c^2 = m_0c^2\left(\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}-1\right) = m_0c^2\left[(1-x)^{-\frac{1}{2}}-1\right].$$

$$|R_1(x)| \le \frac{M}{2!}x^2,\ M = \max|f''(x)|.$$

$$\Rightarrow |f''(x)| = \frac{3m_0c^2}{4\left(1-{}^{v^2}\!/\!_{c^2}\right)^{\frac{5}{2}}} \le \frac{3m_0c^2}{4\left(1-{}^{100^2}\!/\!_{c^2}\right)^{\frac{5}{2}}} < \left(4.17\times 10^{-10}\right)m_0$$