§11.10 Taylor and Maclaurin Series

Question：

i. How to find a power series representation at a of a function f in general?
ii. Will such power series representation equal to the function f itself?

∗General Method：If $f(x)$ has a power series representation at a, say

$$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n,\ |x-a|<R$$

then $c_n=\dfrac{f^{(n)}(a)}{n!}$.

Definition：

i. Taylor series of f at $x = a$ is defined to be $\displaystyle\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$.
ii. If $a = 0$, the corresponding Taylor series is called Maclaurin series.
iii. $\displaystyle\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the n th-degree Taylor polynomial of f at a.

Remark：

$f(x)$ is not necessarily equal to its Taylor series.

Example 1：

Find the Maclaurin series of $f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x\neq 0. \\ 0 & \text{if } x=0. \end{cases}$

Solution：

$\Rightarrow f'(0)=f''(0)=\ldots=f^{(n)}(0)=0$

$\Rightarrow$ The Maclaurin series of $f(x)=0$, but $f(x)\neq 0$ whenever $x\neq 0$.

Example 2：

Let $f(x) = e^x$. Find its Maclaurin series.

Solution：

$$\text{M.S.}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}.\ (\because\ f^{(n)}(x)=e^x).$$

Example 3：

Let $f(x) = \sin x$. Find its Maclaurin series.

Solution：

$$\begin{aligned} f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f^{(3)}(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \end{aligned} \Rightarrow \text{M.S.} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Example 4：

Find the Maclaurin series of $f(x) = \cos x$.

Solution：

$$\text{M.S.} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}.$$

＊Taylor Inequality(誤差估計)：

Let $f(x) - T_n(x) = R_n(x)$, where $T_n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-a)^k$ is called the n th-degree Taylor polynomial of f at a.

If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$,

then $\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$ for $\left|x-a\right| \le d$.

Example 5：

Prove $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for any x.

Solution：

$$\left|R_n(x)\right| \le \frac{e^d}{(n+1)!}|x|^{n+1} \to 0 \text{ as } n \to \infty \text{ for any } x.$$

＊Summary：

i. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$

ii. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

iii. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

iv. $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

＊牛頓推廣 binomial 的展開，即將$(a + b)^k$, $k \in N$ 的展開，推廣至$(a + b)^k$, $k \in R$.

Theorem：

If $k \in R, |x| < 1$, then $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$,

$$\binom{k}{n} = \frac{(k)(k-1)\ldots(k-n+1)}{n!} \ (n \geq 1) \text{ and } \binom{k}{0} = 1.$$

Remark：

目前我們學到 3 個方法的 Taylor series or power series representation.

(i) $\frac{1}{1-x}$ (ii) $c_n = \frac{f^n(a)}{n!}$ (iii) 多項式展開.

Example 6：

$$f(x) = \frac{1}{\sqrt{4-x}}$$

Solution：

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{4-x}} \\
&= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n} \times \left(-\frac{x}{4}\right)^n \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n \times 1 \times 3 \times \ldots \times (2n-1)}{2^n \times n!} \times (-1)^n \frac{x^n}{2^{2n}} \\
&= \sum_{n=0}^{\infty}\frac{1 \times 3 \times \ldots \times (2n-1)}{2^{3n+1} \times n!} x^n
\end{aligned}$$

$$\left(\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}-1\right) \times \ldots \times \left(-\frac{1}{2}-n+1\right)}{n!}\right)$$

Example 7：

Let $f(x) = (1 + x^2)^{\frac{1}{2}}$. Find $f^{(10)}(0) = ?$

Solution：

$$f(x) = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x^2)^n \quad \text{(多項式展開)}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \text{(Maclaurin Series)}$$

$$\Rightarrow k = 10 \Rightarrow n = 5$$

$$\Rightarrow \frac{f^{(10)}(0)}{10!} = \binom{\frac{1}{2}}{5} = \frac{\frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) \times \left(-\frac{5}{2}\right) \times \left(-\frac{7}{2}\right)}{5!} = \frac{7}{2^8}$$

$$\Rightarrow f^{(10)}(0) = \frac{7}{2^8} 10!$$

Example 8：

Let $f(x) = (1 + x^3)^{-\frac{1}{2}}$. Find $f^{(9)}(0) = ?$

Solution：

$$f(x) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (x^3)^n$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$\Rightarrow k = 9 \Rightarrow n = 3$$

$$\Rightarrow \frac{f^{(9)}(0)}{9!} = \binom{-\frac{1}{2}}{3} = \frac{\left(-\frac{1}{2}\right) \times \left(-\frac{3}{2}\right) \times \left(-\frac{5}{2}\right)}{3!} = -\frac{5}{2^4}$$

$$\Rightarrow f^{(9)}(0) = \frac{-5}{2^4} 9!$$

Example 9：

$$3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+...=?$$

Solution：

$$3+\frac{9}{2!}+\frac{27}{3!}+\frac{81}{4!}+...=e^3-1.$$

Example 10：

$$\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}-\frac{1}{8!}+...=?$$

Solution：

$$\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}-\frac{1}{8!}+...=1-\cos 1.$$

Example 11：

$$\int_0^{0.1}\frac{\ln(1+t)}{t}\,dt=?$$

Example 12：

Find the coefficient of x^5 in the Maclaurin series for $f(x)=\int\cos\left(x^2\right)dx.$

Example 13：

Let $g(x)=\cos\left(x^2\right)$. Find $g^{(8)}(0)$.

Example 14：

Find the first three nonzero terms in the Maclaurin series for the function

$f(x)=\cos^{-1}(x).$