§11.1 Sequences

Definition：

- A sequence is a function whose domain is N. (N is the set of all positive integers.)
- In notation, we write $f(n) = a_n$.

Question：Convergence or divergence of $\{a_n\}$

Example 1：

Convergence or divergence?

i.	$a_n = \frac{1}{n}$	Conv.	v.	$a_n = n\sin\left(\frac{\pi}{2n}\right)$	Conv.
ii.	$a_n = (-1)^n$	Div.	vi.	$a_n = \ln(n) - \ln(n+1)$	Conv.
iii.	$a_n = n$	Div.	vii.	$a_n = \left(1 - \frac{2}{n}\right)^n$	Conv.
iv.	$a_n = \cos\left(\frac{n\pi}{2}\right)$	Div.	viii.	$a_n = \sqrt{n^2+1} - n$	Conv.

Theorem 1：

$\lim_{x\to\infty} f(x) = L, x \in R \Rightarrow \lim_{n\to\infty} f(n) = L, n \in N$. (The converse is not true!)

Theorem 2：

$\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 3：

$\{r^n\}_{n=1}^{\infty}$ converges iff $-1 < r \le 1$.

Example 2：

Convergence or divergence? $a_n = \dfrac{\ln n}{n}$.

Solution：

$$\lim_{n\to\infty}\frac{\ln n}{n} = \lim_{n\to\infty}\frac{\frac{1}{n}}{n} = 0 \qquad \text{(X)}$$

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{x} = 0 \qquad \text{(O)}$$

$$\Rightarrow \text{By Theorem 1} \Rightarrow \lim_{n\to\infty}\frac{\ln n}{n} = 0$$

＊註記：

i. 在離散點上講微分是無意義的.

ii. $n^n >> n! >> e^n >> n^3 >> \ln n$ when n is large.

Theorem 4：Let $\{a_n\}$ be monotonic(單調) and bounded(有界) $\Rightarrow \{a_n\}$ converges.

Example 3：

Prove that $\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+}...}}$ exist.

Proof：

Let $a_1 = \frac{1}{4}, a_{n+1} = \frac{1}{4+a_n}$.

(i). Clearly, a_n is monotonic decreasing.

(ii). Moreover, $a_n \geq 0$ for all $n \in N$.

1. $a_1 \geq 0$.
2. Suppose $a_n \geq 0$, then $a_{n+1} = \frac{1}{4+a_n} \geq \frac{1}{4+0} = \frac{1}{4}$.

By applying Theorem 4, we see that $\lim_{n\to\infty} a_n$ has a limit, say x.

$$x = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{4+a_n}.$$

That is, $x = \frac{1}{4+x} \Rightarrow x = -2 \pm \sqrt{5}$ (負不合，因為$x \geq 0$)