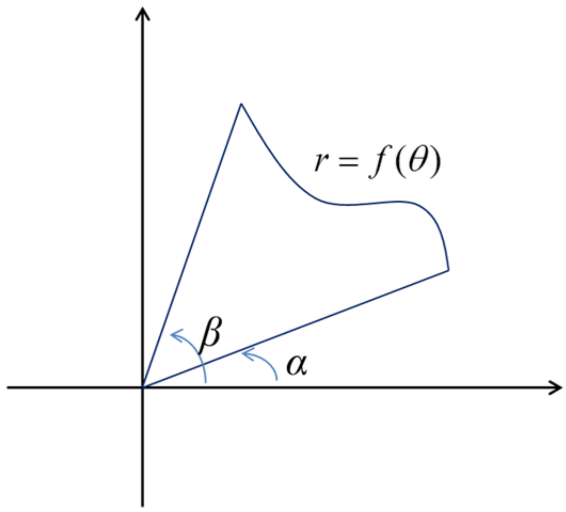
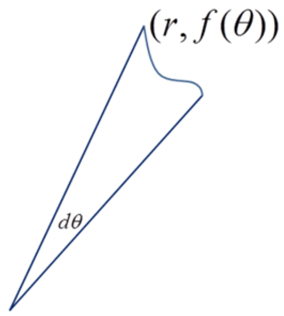
§10-4 Area and Lengths in Polar Coordinates

I. Areas





$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} f^2(\theta) d\theta.$$

II. Arc Length

$$x = f(\theta)\cos\theta \quad y = f(\theta)\sin\theta$$

$$L = \int dl = \int \sqrt{(dx)^2 + (dy)^2} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

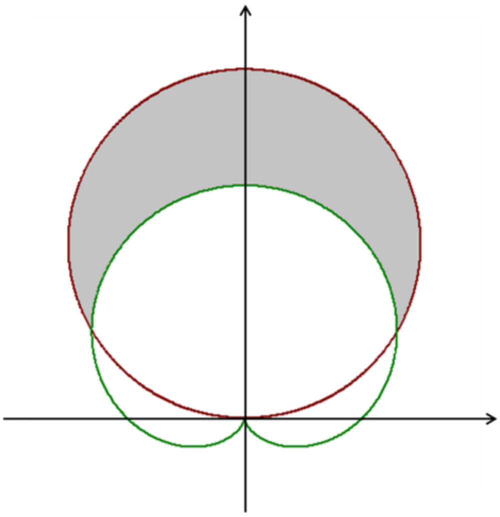
III. Surface Area

$r = f(\theta), \alpha \le \theta \le \beta,$ around the $x-$axis

$$S = \int 2\pi y dl = \int_{\alpha}^{\beta} 2\pi \underbrace{f(\theta)\sin\theta}_{y} \underbrace{\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}_{ds} \cdot$$

Example 1：Find the area of the region that lies inside the circle r = 3sinθ and outside the cardioid r = 1 + sinθ.

Solution：

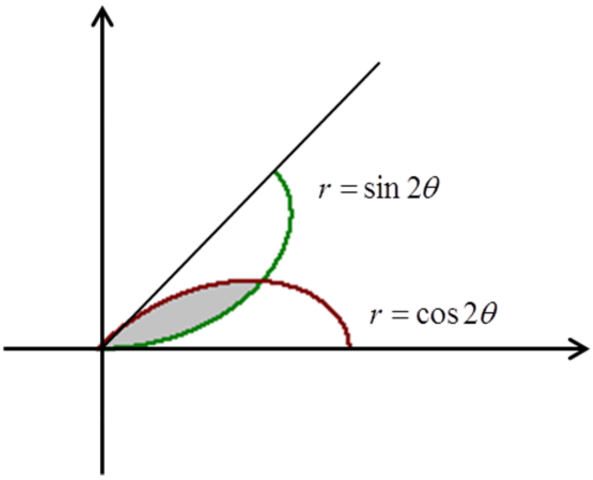


$$\begin{cases} r = 3\sin\theta \\ r = 1+\sin\theta \end{cases} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$A = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left[(3\sin\theta)^2 - (1+\sin\theta)^2\right] d\theta$$
$$= \pi$$

Example 2： Find the area of the region that lies inside both curves r = sin2θ, r = cos2θ.

Solution：



$r=\cos 2\theta,$ 4-leaves rose

$$r=\sin 2\theta=\cos\left(\frac{\pi}{2}-2\theta\right)=\cos 2\left(\frac{\pi}{4}-\theta\right)=\cos 2\left(\theta-\frac{\pi}{4}\right)$$

$$\begin{cases} r=\sin 2\theta \\ r=\cos 2\theta \end{cases}\Rightarrow \tan 2\theta=1\Rightarrow \theta=\frac{\pi}{8}.$$

灰色區域面積 $=\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\frac{1}{2}\cos^2 2\theta d\theta.$

$$=\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\frac{1+\cos 4\theta}{4}d\theta$$

$$=-\left(\frac{\theta}{4}+\frac{\sin 4\theta}{16}\right)\Bigg|_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$=\left(\frac{\pi}{32}-\frac{1}{16}\right).$$

$\Rightarrow$ 總面積 $=16$ 個灰色區域 $=16\left(\frac{\pi}{32}-\frac{1}{16}\right)=\frac{\pi}{2}-1.$

Example 3：Find the length of cardioid r = 1 + sin*θ*.

Solution：

$$L = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2+2\sin\theta}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{4-4\sin^2\theta}{2-2\sin\theta}}\, d\theta$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{2-2\sin\theta}}\, d\theta \qquad u = 2-2\sin\theta,\ \ du = -2\cos\theta d\theta$$

$$= 2\int_{0}^{4} \frac{du}{\sqrt{u}} = 4u^{\frac{1}{2}}\Big|_0^4 = 8.$$