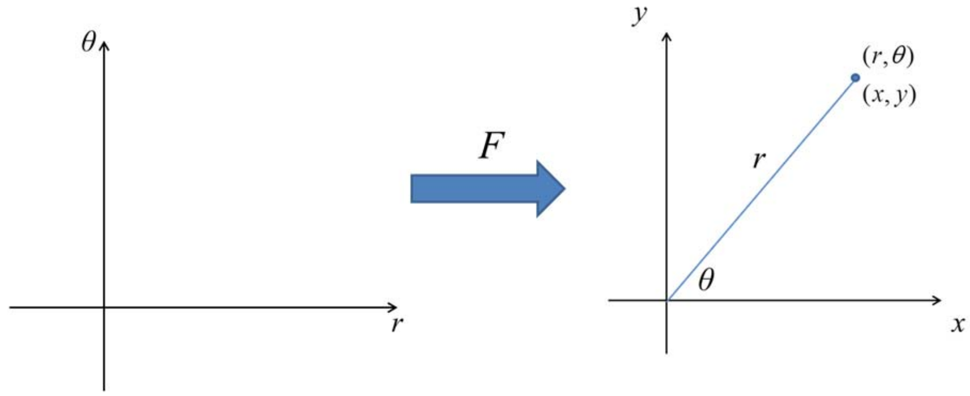
§10-3 Polar Coordinates

I.

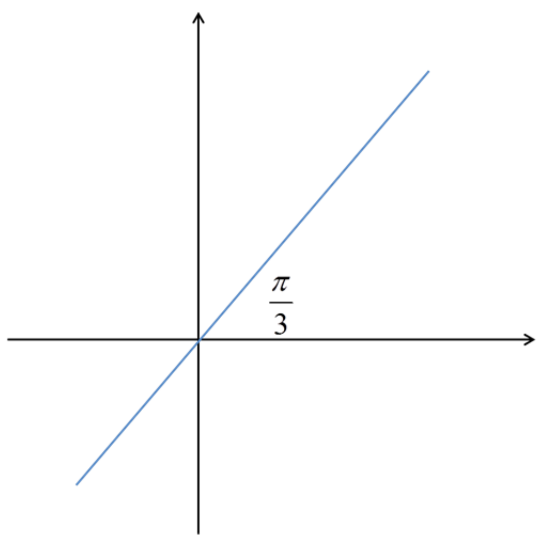


$$F(r,\theta)=(r\cos\theta, r\sin\theta)=(x,y)$$

<table>
<tr><td colspan="2">$(x,y)\leftrightarrow(r,\theta)$</td></tr>
<tr><td>$x=r\cos\theta$
$y=r\sin\theta$</td><td>$r^2=x^2+y^2$
$\tan\theta=\dfrac{y}{x}$</td></tr>
<tr><td colspan="2">i. r may be negative.
ii. 同一點(r ,θ)的表示法不唯一
$\left(-1,\dfrac{\pi}{4}\right)=\left(1,\dfrac{5\pi}{4}\right)=\left(1,\dfrac{13\pi}{4}\right)$</td></tr>
</table>

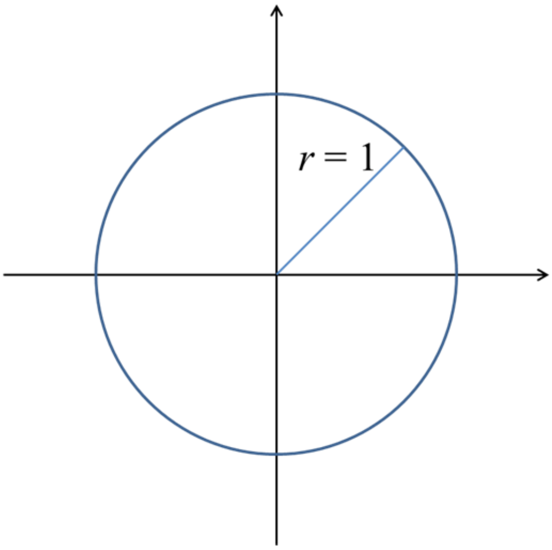
Example 1： $\theta=\dfrac{\pi}{3}$ means a straight line.

Solution：

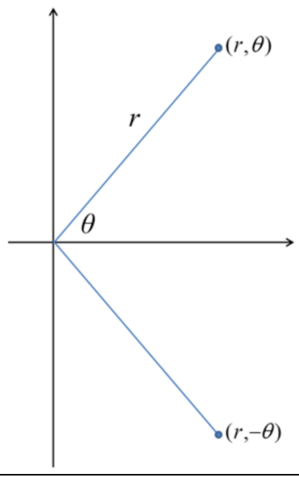
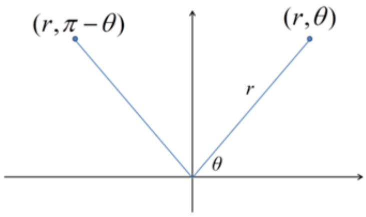
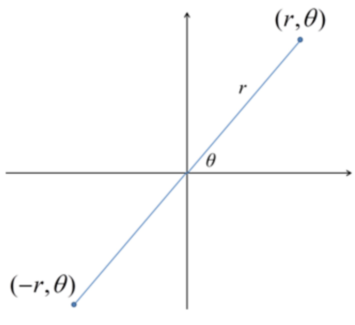


Example 2：$r = 1$ means a circle.

Solution：



II. Symmetry

對稱 x 軸	對稱 y 軸	對稱原點
θ 用 $-\theta$ 代 方程式不變	θ 用 $\pi-\theta$ 代 方程式不變	r 用 $-r$ 代 方程式不變
		

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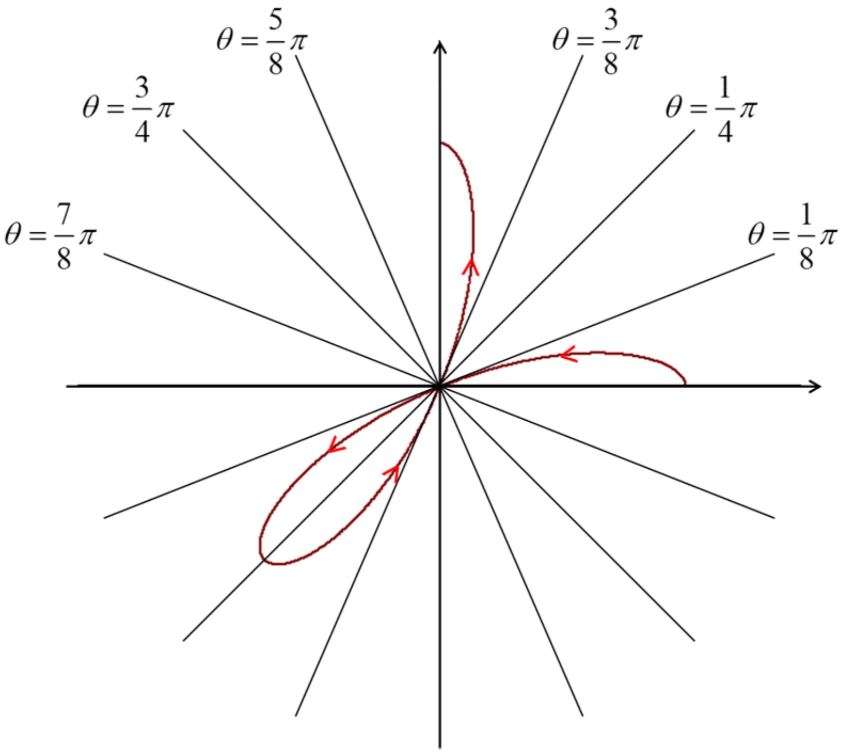
Example 3： Sketch $r = 2\cos 4\theta$

Solution：

對稱 x 軸和 y 軸



上圖是 θ 從 $0 \to \frac{\pi}{8} \to \frac{2\pi}{8} \to \frac{3\pi}{8} \to \frac{4\pi}{8}$ (即第一象限)的 r 的變化圖(依箭頭方向)。

將此箭頭圖 reflect on the x-axis and y-axis，我們得到一個 8-leaves rose 的圖。

III. Tangents to Polar Curves

Polar Curve：$r = f(\theta)$

Its corresponding parametric equation：$x = f(\theta)\cos\theta,\ y = f(\theta)\sin\theta.$

Slope of the tangent：$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$

Remark：

i. Horizontal tangents：points at which $\frac{dy}{d\theta}=0$.

ii. Vertical tangents：points at which $\frac{dx}{d\theta}=0$.

Example 4：

i. Sketch $r = 1 + \sin\theta$.

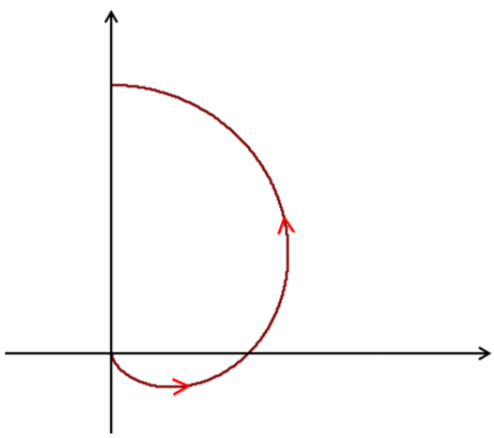
ii. Find $\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}$.

iii. Find the points on the curve r = 1 + sinθ where the tangent line is horizontal or vertical.

Solution：

i. 對稱 y 軸，考慮 $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ 的部分

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
r	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2



ii. $$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}\sin\theta+r\cos\theta}{\frac{dr}{d\theta}\cos\theta-r\sin\theta}=\frac{\cos\theta(1+2\sin\theta)}{(1+\sin\theta)(1-2\sin\theta)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = -1.$$

iii. $$\frac{dy}{d\theta} = \cos\theta(1+2\sin\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

$$\frac{dx}{d\theta} = (1+\sin\theta)(1-2\sin\theta) \Rightarrow \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$